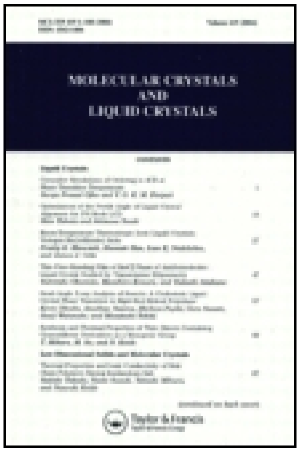

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A Unified Phenomenology of Nematic Liquid Crystals

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A Unified Phenomenology of Nematic Liquid Crystals

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Liquid crystals are well characterized by the orientational order parameter. In most cases, the orientational order parameter of nematic liquid crystals is uniaxial and is expressed by a traceless symmetric tensor of rank two. We have developed a phenomenological theory whose starting point is the free energy expanded in powers of the orientational order parameter. By introducing the surface energy, several phenomena, including wall effects, can be explained in a unified manner. Hydrodynamic phenomena are also discussed.

I. INTRODUCTION

After deGennes introduced the orientational order parameter in order to explain the light scattering of nematic liquid crystals,[1] several phenomena have been described by the continuum theory, such as the nematic–isotropic phase transition,[1] director deformations,[2] hydrodynamics,[1] and so on. The nematic state can be characterized by the orientational order parameter, which is expressed by a traceless symmetric tensor of rank two. When one discusses smectic and/or re-entrant nematic[3] liquid crystals, it is necessary to consider translational order parameters and density as well. However, such order parameters vanish in ordinary nematic liquid crystals.

On the other hand, the development of practical applications to display devices has generated several fundamental problems: 1) tilted alignment at solid surfaces,[4] 2) temperature dependence of tilt angle,[5] 3) abnormal gradient of the director at the surface,[6] 4) temperature behaviour of the order parameter at the surface,[7] and so on. These problems are all concerned with wall effects. The phenomenology developed by deGennes can explain the bulk phenomena, but needs to be extended to describe the wall effects.

In order to describe the wall effects, the author has introduced the surface order parameter.[8] Several surface phenomena, e.g. temperature dependence of the tilt angle,[8] abnormal gradients of the director,[9] and wall effects on Freederiksz transitions,[10] can be successfully described by taking into account the surface free energy expanded in terms of the surface order parameter.

In this paper, we will reconstruct a unified phenomenology for nematic liquid crystals. Our starting point is the total free energy of the system under consideration. We assume that nematic liquid crystals have uniaxial symmetry. Further, we neglect the smectic, cholesteric, and re-entrant liquid crystals, but these will be mentioned briefly in the discussion.

II. PHENOMENOLOGY

Since nematic liquid crystals have uniaxial symmetry and the symmetry disappears at temperatures above the clearing point, the orientational order parameter is given by a traceless symmetric tensor of rank two, Q_{ij}. We assume that the free energy is expanded in powers of Q_{ij}. First of all, the fundamental equations of the static equilibrium state will be discussed, and then those for the dynamical properties will be described.

1. Static equilibrium

1.1 Total free energy. One expects the free energy per unit volume to have the following form, in which only terms which have rotational invariance are retained:

$$\Lambda_c = \Lambda_0 + \frac{A}{2} Q_{ij} Q_{ij} + \frac{B}{3} Q_{ij} Q_{jk} Q_{ki} + \frac{C}{4} (Q_{ij} Q_{ij})^2, \qquad (1)$$

where $A = (T - T^*)a$ with T^* the nematic–isotropic transition temperature. The coefficients B and C are constants. Note that repeated indices indicate summation. When the order parameter varies slowly from point to point, the distortion free energy is given by quadratic terms in the gradients of Q_{ij}, i.e.

$$\Lambda_d = \frac{L_1}{2} Q_{ij,k} Q_{ij,k} + \frac{L_2}{2} Q_{ik,i} Q_{jk,j}, \qquad (2)$$

where a comma in the indices means the differentiation with respect to the coordinate appearing after the comma. The coefficients are elastic constants. Furthermore, external fields will be considered. The electric and magnetic field energies are expressed as

$$\Lambda_e = -\tfrac{1}{3}\Delta\alpha Q_{ij}E_iE_j,$$
$$\Lambda_m = -\tfrac{1}{3}\Delta\chi Q_{ij}H_iH_j, \tag{3}$$

where $\Delta\alpha$ and $\Delta\chi$ denote the anisotropy of the permittivity and the magnetic susceptibility of the molecules, respectively. Thus the bulk free energy per unit volume Λ is expressed as follows.

$$\Lambda = \Lambda_c + \Lambda_d + \Lambda_e + \Lambda_m \tag{4}$$

Considering the total free energy of the system, the surface free energy must be expressed in terms of the surface order parameter. The surface order parameter $\overline{Q}_{ij}$ is defined as the order parameter at the surface. The symbol bar means the quantity at the surface; this notation is used throughout the paper. We define the surface as follows: there is a force to anchor the liquid crystal molecules; this force is characterized by an easy axis. The surface free energy per unit area is then as follows, retaining terms through second order in $\overline{Q}_{ij}$:

$$\overline{\Lambda} = \overline{\Lambda}_0 - \tfrac{1}{3}\Delta\beta\overline{Q}_{ij}d_id_j - \tfrac{2}{3}\Delta\gamma\left(\overline{Q}_{ij}d_j\right)^2, \tag{5}$$

where $\Delta\beta$ and $\Delta\gamma$ are the anchoring strengths and have positive values, and $\mathbf{d}$ is the easy axis.

Finally, the total free energy of the system is

$$F = \int\Lambda d^3\mathbf{r} + \int\overline{\Lambda}d^2\mathbf{r}. \tag{6}$$

We restrict our attention to the nematic state. In the nematic phase, the order parameter can be written as

$$Q_{ij} = \tfrac{3}{2}S\left(n_in_j - \tfrac{1}{3}\delta_{ij}\right), \tag{7}$$

where S is the scalar order parameter, which shows the degree of alignment of the liquid crystal molecules. The vector $\mathbf{n}$ stands for the director. Substituting Eq. (7) into Eqs. (1)–(3), the bulk free energy reduces to

$$\begin{aligned}\Lambda = \Lambda_0 &- \tfrac{1}{2}\Delta\alpha S(\mathbf{n}\cdot\mathbf{E})^2 - \tfrac{1}{2}\Delta\chi S(\mathbf{n}\cdot\mathbf{H})^2 + \tfrac{3}{4}AS^2 + \tfrac{1}{4}BS^3 + \tfrac{9}{16}CS^4 \\ &+ \tfrac{1}{2}L_1' S^2(\operatorname{div}\mathbf{n})^2 + \tfrac{1}{2}L_2' S^2(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})^2 + \tfrac{1}{2}L_3' S^2(\mathbf{n}\times\operatorname{rot}\mathbf{n})^2 \\ &+ \tfrac{1}{2}L_1''(\operatorname{grad} S)^2 + \tfrac{1}{2}L_2''(\mathbf{n}\cdot\operatorname{grad} S)^2 \\ &+ \tfrac{1}{2}L_1''' S\operatorname{div}\mathbf{n}\,(\mathbf{n}\cdot\operatorname{grad} S) + \tfrac{1}{2}L_2''' S\operatorname{grad} S(\mathbf{n}\times\operatorname{rot}\mathbf{n}). \end{aligned} \tag{8}$$

The coefficients in eq. (8) are related to those in eqs. (1)–(3) by

$$\begin{aligned} &L_1' = \tfrac{9}{4}L_2, \qquad L_2' = \tfrac{9}{4}(2L_1), \qquad L_3' = \tfrac{9}{4}(2L_1 + L_2), \\ &L_1'' = \tfrac{1}{3}\left(\tfrac{3}{2}L_1 + \tfrac{1}{4}L_2\right), \qquad L_2'' = \tfrac{3}{4}L_2, \qquad L_1''' = 3L_2, \qquad L_2''' = \tfrac{3}{2}L_2 \end{aligned} \tag{9}$$

Similarly, the surface free energy is derived as

$$\bar{\Lambda} = \bar{\Lambda}_0 - \tfrac{1}{2}\Delta\beta S(\mathbf{n}\cdot\mathbf{d})^2 - \tfrac{1}{2}\Delta\gamma S^2(\mathbf{n}\cdot\mathbf{d})^2. \tag{10}$$

Note that the bulk free energy is a function of S, $S_{,i}$, n_i, and $n_{i,j}$ while the surface free energy depends on S and n_i.

1.2 Equilibrium state and torque equations. The system changes its state to minimize the total free energy F. We assume the scalar order parameter S and the director $\mathbf{n}$ are independent of each other. Calculating δF, allowing both for variations in n_i and $n_{i,j}$ and in S and $S_{,i}$, the following four equations are obtained under the condition $\mathbf{n}^2 = 1$:

$$\frac{\partial\Lambda}{\partial S} - \left(\frac{\partial\Lambda}{\partial S_{,i}}\right)_{,i} = 0, \tag{11}$$

$$\frac{\partial\bar{\Lambda}}{\partial S} + \left(\frac{\partial\bar{\Lambda}}{\partial S_{,i}}\right)\nu_i = 0, \tag{12}$$

$$\frac{\partial\Lambda}{\partial n_i} - \left(\frac{\partial\Lambda}{\partial n_{i,j}}\right)_{,j} = \lambda n_i = -h_i, \tag{13}$$

$$\frac{\partial\bar{\Lambda}}{\partial n_i} + \left(\frac{\partial\bar{\Lambda}}{\partial n_{i,j}}\right)\nu_j = \bar{\lambda} n_i = -\bar{h}_i, \tag{14}$$

where $\boldsymbol{\nu}$ is the normal unit vector to the surface, λ is an arbitrary constant (Lagrange multiplier). The vectors $\mathbf{h}$ and $\bar{\mathbf{h}}$ are the molecular fields in the bulk and at the surface, respectively. According to Eqs. (13) and (14), the director $\mathbf{n}$ cannot be determined uniquely. Thus we consider the torque acting on the director and demand that this torque vanish; i.e.

$$\boldsymbol{\Gamma} = \mathbf{n} \times \mathbf{h} = 0 \quad \text{and} \quad \bar{\boldsymbol{\Gamma}} = \mathbf{n} \times \bar{\mathbf{h}} = 0. \tag{15}$$

The equilibrium state is then given by the following four equations corresponding to Eqs. (11), (12), and (15):

$$\begin{aligned} &-\tfrac{1}{2}\Delta\alpha(\mathbf{n}\cdot\mathbf{E})^2 - \tfrac{1}{2}\Delta\chi(\mathbf{n}\cdot\mathbf{H})^2 + \tfrac{3}{2}AS + \tfrac{3}{4}BS^2 + \tfrac{9}{4}CS^3 \\ &\quad + L_1'S(\operatorname{div}\mathbf{n})^2 + L_2'S(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})^2 + L_3'S(\mathbf{n}\times\operatorname{rot}\mathbf{n})^2 \\ &\quad - L_1''\operatorname{div}(\operatorname{grad} S) - L_2''\mathbf{n}\cdot(\operatorname{grad} S)\operatorname{div}\mathbf{n} \\ &\quad + \tfrac{1}{2}L_1'''\{\operatorname{div}\mathbf{n}(\mathbf{n}\cdot\operatorname{grad} S) - S\operatorname{div}(\mathbf{n}\operatorname{div}\mathbf{n})\} \\ &\quad - \tfrac{1}{2}L_2'''\{\operatorname{grad} S(\mathbf{n}\times\operatorname{rot}\mathbf{n}) - S\operatorname{div}(\mathbf{n}\times\operatorname{rot}\mathbf{n})\} = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &-\left(\tfrac{1}{2}\Delta\beta + \Delta\gamma\bar{S}\right)(\bar{\mathbf{n}}\cdot\mathbf{d})^2 + L_1''\overline{\operatorname{grad} S}\cdot\boldsymbol{\nu} + L_2''\{\bar{\mathbf{n}}(\bar{\mathbf{n}}\cdot\overline{\operatorname{grad} S})\}\boldsymbol{\nu} \\ &\quad + \tfrac{1}{2}L_1'''\bar{S}(\overline{\operatorname{div}\mathbf{n}})\bar{\mathbf{n}}\cdot\boldsymbol{\nu} - \tfrac{1}{2}L_2'''\bar{S}\bar{\mathbf{n}}\cdot(\boldsymbol{\nu}\times\overline{\operatorname{rot}\mathbf{n}}) = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} &-L_1'S^2\mathbf{n}\times\operatorname{grad}(\operatorname{div}\mathbf{n}) + L_2'S^2\mathbf{n}\times\big[(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})\operatorname{rot}\mathbf{n} \\ &\qquad\qquad\qquad\qquad + \operatorname{rot}\{(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})\mathbf{n}\}\big] \\ &\quad - L_3'S^2\mathbf{n}\times\big[(\mathbf{n}\times\operatorname{rot}\mathbf{n})\times\operatorname{rot}\mathbf{n} + \operatorname{rot}\{\mathbf{n}\times(\mathbf{n}\times\operatorname{rot}\mathbf{n})\}\big] \\ &\quad + L_2''(\mathbf{n}\cdot\operatorname{grad} S)(\mathbf{n}\times\operatorname{grad} S) - \tfrac{1}{2}L_1'''S\mathbf{n} \\ &\quad \times\{\operatorname{div}\mathbf{n}\operatorname{grad} S - \operatorname{grad}(\mathbf{n}\cdot\operatorname{grad} S)\} \\ &\quad + \tfrac{1}{2}L_2'''S\mathbf{n}\times\{2\operatorname{grad} S\times\operatorname{rot}\mathbf{n} + \mathbf{n}\operatorname{div}(\operatorname{grad} S)\} \\ &\quad - \Delta\alpha S(\mathbf{n}\cdot\mathbf{E})(\mathbf{n}\times\mathbf{E}) - \Delta\chi S(\mathbf{n}\cdot\mathbf{H})(\mathbf{n}\times\mathbf{H}) = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} &L_1'\bar{S}^2\bar{\mathbf{n}}\times(\overline{\operatorname{div}\mathbf{n}})\boldsymbol{\nu} - L_2'\bar{S}^2\bar{\mathbf{n}}\times(\overline{\mathbf{n}\cdot\operatorname{rot}\mathbf{n}})\bar{\mathbf{n}}\times\boldsymbol{\nu} + L_3'\bar{S}^2\bar{\mathbf{n}}\times(\overline{\mathbf{n}\times\operatorname{rot}\mathbf{n}}) \\ &\quad \times\bar{\mathbf{n}}\times\boldsymbol{\nu} - \tfrac{1}{2}L_1'''\bar{S}(\overline{\mathbf{n}\cdot\operatorname{grad} S})\bar{\mathbf{n}}\times\boldsymbol{\nu} - \tfrac{1}{2}L_2'''\bar{S}\bar{\mathbf{n}}\times\boldsymbol{\nu}\,\overline{\operatorname{grad} S} \\ &\quad - \left(\Delta\beta\bar{S} + \Delta\gamma\bar{S}^2\right)(\bar{\mathbf{n}}\cdot\mathbf{d})(\bar{\mathbf{n}}\times\mathbf{d}) = 0 \end{aligned} \tag{19}$$

The equations (18) and (19) are the torque equations, i.e. Eq. (15), in the bulk and at the surface. These four equations yield the equilibrium values of S and $\mathbf{n}$ in the nematic state. The explicit solutions of S and $\mathbf{n}$ cannot be obtained analytically. We will discuss the solution for several special cases from now on.

1.3 Discussion. Let us now consider several special cases. The magnetic field is neglected because the behaviour of liquid crystals when a magnetic field is applied is similar to that when an electric field is applied.

A. $\mathbf{E} = 0$, $\mathbf{n} = \text{const.}$, and $S = \text{const.}$

This is the most simple case, corresponding to a uniform nematic 'single crystal'. In this case, the Eq. (18) becomes identically equal to zero, and Eqs. (16), (17), and (19) reduce to

$$\tfrac{3}{2}AS + \tfrac{3}{4}BS^2 + \tfrac{9}{4}CS^3 = 0, \tag{20}$$

$$\left(\tfrac{1}{2}\Delta\beta + \Delta\gamma\bar{S}\right)(\bar{\mathbf{n}} \cdot \bar{\mathbf{d}})^2 = 0, \tag{21}$$

$$\left(\Delta\beta\bar{S} + \Delta\gamma\bar{S}^2\right)(\bar{\mathbf{n}} \cdot \bar{\mathbf{d}})(\bar{\mathbf{n}} \times \bar{\mathbf{d}}) = 0. \tag{22}$$

From Eqs. (21) and (22), the solution for the director is $\bar{\mathbf{n}} \equiv \mathbf{n} \perp \mathbf{d}$. This solution yields the maximum free energy by eq. (10). Thus the anchoring strength $\Delta\beta$ and $\Delta\gamma$ are both zero in order to satisfy eqs. (21) and (22). This means there is no force to anchor the liquid crystal molecules, i.e., only the bulk value of S is obtained. Equation (20) agrees with the Landau-deGennes expansion in describing the nematic–isotropic phase transition in the bulk liquid.[1] This result is not new. But taking into account the wall effect, this case can only exist for infinite sample volume. The wall effect will be treated in case B below.

B. $\mathbf{E} = 0$, $\mathbf{n} =$ const., and $S \neq$ const.

This case also corresponds to a nematic 'single crystal', but the degree of alignment can vary spatially. The terms involving $\mathbf{E}$ and $n_{i,j}$ vanish and Eqs. (16)–(19) reduce to

$$\tfrac{3}{2}AS + \tfrac{3}{4}BS^2 + \tfrac{9}{4}CS^3 - L_1''\mathrm{div}(\mathrm{grad}\, S) - L_2''\mathrm{div}\{\mathbf{n}(\mathbf{n} \cdot \mathrm{grad}\, S)\} = 0, \tag{23}$$

$$-\left(\tfrac{1}{2}\Delta\beta + \Delta\gamma\bar{S}\right)(\bar{\mathbf{n}} \cdot \mathbf{d})^2 + L_1''\overline{\mathrm{grad}\, S} \cdot \boldsymbol{\nu} + L_2''\left\{\bar{\mathbf{n}}\left(\overline{\mathbf{n} \cdot \mathrm{grad}\, S}\right)\right\}\boldsymbol{\nu} = 0, \tag{24}$$

$$L_2''(\mathbf{n} \cdot \mathrm{grad}\, S)(\mathbf{n} \times \mathrm{grad}\, S) = 0, \tag{25}$$

$$\begin{aligned} &-\tfrac{1}{2}L_1'''\bar{S}\left(\overline{\mathbf{n} \cdot \mathrm{grad}\, S}\right)\mathbf{n} \times \boldsymbol{\nu} - \tfrac{1}{2}L_1'''\bar{S}(\bar{\mathbf{n}} \times \boldsymbol{\nu})\overline{\mathrm{grad}\, S} \\ &+\left(\Delta\beta\bar{S} + \Delta\gamma\bar{S}^2\right)(\bar{\mathbf{n}} \cdot \bar{\mathbf{d}})(\bar{\mathbf{n}} \times \bar{\mathbf{d}}) = 0. \end{aligned} \tag{26}$$

The director at the surface $\bar{\mathbf{n}}$ is determined by Eq. (26). Since $L_2'' > 0$, the director must be perpendicular to grad S from Eqs. (8) and (25). Then Eqs. (23) and (24) simplify to

$$\tfrac{3}{2}AS + \tfrac{3}{4}BS^2 + \tfrac{9}{4}CS^3 - L_1''\mathrm{div}(\mathrm{grad}\, S) = 0, \tag{27}$$

$$-\left(\tfrac{1}{2}\Delta\beta + \Delta\gamma\bar{S}\right) + L_1''\overline{\mathrm{grad}\, S} \cdot \boldsymbol{\nu} = 0. \tag{28}$$

Equation (27) agrees with the equation derived by Sheng,[12] who solved it with $\bar{S} = 1$, and obtained the temperature and spatial dependences of the order parameter S. However the order parameter at the surface $\bar{S}$ is not unity in general[7]: Akahane and Tako[13] simulated the spatial and temperature dependences of S for a given value of $\bar{S}$ not equal to unity. Introducing the surface energy, the order parameter is changed by the anchoring strength at the surface, i.e. the order parameter of nematic liquid crystals is governed by the anchoring strength at the treated solid surface as well as intermolecular interaction. This phenomenon was pointed out by Akahane and Tako.[13]

Let us evaluate the order parameter at the surface. The surface is taken to be in the x-y plane and $\boldsymbol{v}$ is parallel to the z-axis. We use the following abbreviations for simplicity:

$$f(S) = \tfrac{3}{2}AS + \tfrac{3}{4}BS^2 + \tfrac{9}{4}CS^3, \text{ and } \ F(S) = \tfrac{3}{4}AS^2 + \tfrac{1}{4}BS^3 + \tfrac{9}{16}CS^4.$$

Equations (27) and (28) can be rewritten as

$$f(S) - L_1'' \frac{d^2S}{dz^2} = 0, \tag{29}$$

$$-\left(\frac{1}{2}\Delta\beta + \Delta\gamma\bar{S}\right) + L_1'' \frac{d\bar{S}}{dz} = 0. \tag{30}$$

Integrating Eq. (29) with respect to z and using $dS/dz = 0$ at the center of the cell, one obtains the following formula:

$$F(\bar{S}) - F(S_m) = \frac{1}{2L_1''}\left(\frac{1}{2}\Delta\beta + \Delta\gamma\bar{S}\right)^2, \tag{31}$$

where S_m is the order parameter at the center. From Eqs. (10) and (31), the surface free energy per unit area is then given by

$$\bar{\Lambda}(\bar{S}) = \bar{\Lambda}_0 - \Delta\beta\Delta\gamma\bar{S} + \tfrac{3}{4}\overline{AS}^2 + \tfrac{1}{4}\overline{BS}^3 + \tfrac{9}{16}\overline{CS}^4. \tag{32}$$

Finally the order parameter $\bar{S}$ is obtained by solving the following equation:

$$-\Delta\beta\Delta\gamma + \tfrac{3}{2}\overline{AS} + \tfrac{3}{4}\overline{BS}^2 + \tfrac{9}{4}\overline{CS}^3 = 0. \tag{33}$$

This equation is discussed by Mada *et al.*[14] They derived it by a direct expansion in the scalar order parameter S, whereas the result is derived here by rather more unified development.

The coefficient $\bar{A}$ is given by

$$\bar{A} = (T - \bar{T}^*)\bar{a} = 2L_1''\left\{T - \left(T^* + \frac{2\Delta\gamma^2}{3L_1''a}\right)a\right\}, \tag{34}$$

where $\bar{T}^*$ is the surface transition temperature. According to several references: $a \sim 10^{-2}$ J/cm^3 K,[15] $\Delta\gamma \sim 10^{-2}$ erg/cm^2,[16] and $L_1'' \sim 10^{-7}$ dyn[17]; hence $2\Delta\gamma^2/3L_1''a \sim 1$ K. The phase transition at the surface is higher than the bulk transition by about 1 K. This agrees qualitatively with experimental results.[18]

C. $E \neq 0$, $\mathbf{n} \neq$ const., and $S =$ const.

This is the case of director deformation when an electric field is applied. As the order parameter is constant throughout the system, we put the coefficients in conventional notation: $L_i'S^2 = k_i$ ($i = 1$, 2, and 3), $\Delta\beta\bar{S} + \Delta\gamma\bar{S}^2 = \Delta\pi$, $\Delta\beta/2 + \Delta\gamma\bar{S} = \Delta\pi'$, and $\Delta\alpha S = \Delta\epsilon$. Then Eqs. (16)–(19) reduce to

$$-\tfrac{1}{2}\Delta\epsilon(\mathbf{n}\cdot\mathbf{E})^2 + Sf(S) + k_1(\operatorname{div}\mathbf{n})^2 + k_2(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})^2 + k_3(\mathbf{n}\times\operatorname{rot}\mathbf{n})^2 = 0, \tag{35}$$

$$\tfrac{1}{2}L_1'''\bar{S}(\overline{\operatorname{div}\mathbf{n}})\bar{\mathbf{n}}\cdot\boldsymbol{\nu} + \tfrac{1}{2}L_2'''\bar{S}\bar{\mathbf{n}}\cdot(\boldsymbol{\nu}\times\overline{\operatorname{rot}\mathbf{n}}) - \Delta\pi'(\bar{\mathbf{n}}\cdot\mathbf{d})^2 = 0, \tag{36}$$

$$\begin{aligned}&-k_1\mathbf{n}\times\operatorname{grad}(\operatorname{div}\mathbf{n}) + k_2\mathbf{n}\times[(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})\operatorname{rot}\mathbf{n} + \operatorname{rot}\{(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})\mathbf{n}\}]\\ &\quad -k_3\mathbf{n}\times[(\mathbf{n}\times\operatorname{rot}\mathbf{n})\times\operatorname{rot}\mathbf{n} + \operatorname{rot}\{\mathbf{n}\times(\mathbf{n}\times\operatorname{rot}\mathbf{n})\}]\\ &\quad -\Delta\epsilon(\mathbf{n}\cdot\mathbf{E})(\mathbf{n}\times\mathbf{E}) = 0,\end{aligned} \tag{37}$$

$$\begin{aligned}&\bar{k}_1\bar{\mathbf{n}}\times(\overline{\operatorname{div}\mathbf{n}})\boldsymbol{\nu} - \bar{k}_2\bar{\mathbf{n}}\times(\overline{\mathbf{n}\cdot\operatorname{rot}\mathbf{n}})\bar{\mathbf{n}}\times\boldsymbol{\nu} + \bar{k}_3\bar{\mathbf{n}}\times(\overline{\mathbf{n}\times\operatorname{rot}\mathbf{n}})\\ &\quad \times\bar{\mathbf{n}}\times\boldsymbol{\nu} - \Delta\pi(\bar{\mathbf{n}}\cdot\mathbf{d})(\bar{\mathbf{n}}\times\mathbf{d}) = 0.\end{aligned} \tag{38}$$

The last two equations, (37) and (38), were derived by Mada.[8]

Equations (37) and (38) lead to many of the surface phenomena mentioned in the introduction. Solving Eqs. (35)–(38), not only the director distribution but also the order parameter S can be obtained.

2. Dynamical properties

We now consider the wall effects on *dynamical* phenomena. This is the first attempt along these lines of which we are aware.

2.1 Dissipation. Until now, we have discussed the static equilibrium state. When the system undergoes a transition from one state to another state by external force, the dissipative energy must be considered. For simplicity, we restrict our attention to isothermal processes. The total dissipation $T\dot{s}$ is equal to the decrease in stored free energy per unit time, where T is the temperature, s is the entropy, and the dot signifies the derivative with respect to time. Thus,

$$T\dot{s} = -\frac{d}{dt}\int\left(\frac{1}{2}\rho \mathbf{v}^2 + \Lambda\right) d^3\mathbf{r} - \frac{d}{dt}\int \overline{\Lambda}\, d^2\mathbf{r}, \qquad (39)$$

where ρ is the density of the liquid crystal, and $\mathbf{v}$ is the velocity of the fluid. Let us now evaluate this expression explicitly. The first term is given by the following formula derived using the equations of motion and integrating by parts:

$$-\frac{d}{dt}\int \frac{1}{2}\rho \mathbf{v}^2 d^3\mathbf{r} = \int \sigma_{ij} v_{j,i}\, d^3\mathbf{r} - \int \sigma_{ij} v_j \nu_i\, d^2\mathbf{r}, \qquad (40)$$

where σ_{ij} is the stress tensor. The second term is derived as follows, including the surface term[19]:

$$\begin{aligned} -\frac{d}{dt}\int \Lambda\, d^3\mathbf{r} &= \int\left(-\sigma^e_{ij} v_{j,i} + h_i \dot{n}_i\right) d^3\mathbf{r} \\ &\quad - \int\left(\frac{\partial\overline{\Lambda}}{\partial n_{i,j}} \nu_j \dot{n}_i + \frac{\partial\overline{\Lambda}}{\partial S_{,i}} \nu_i \dot{S}\right) d^2\mathbf{r}, \end{aligned} \qquad (41)$$

where σ^e_{ij} is the Ericksen stress. The final term, the surface contribution, is

$$-\frac{d}{dt}\int \overline{\Lambda} d^2\mathbf{r} = \int\left(-\frac{\partial\overline{\Lambda}}{\partial n_i}\dot{n}_i - \frac{\partial\overline{\Lambda}}{\partial S}\dot{S}\right) d^2\mathbf{r}. \qquad (42)$$

Combining all these contributions, the total dissipation can be written as follows:

$$T\dot{s} = \int \left(\sigma'_{ij} v_{j,i} + h_i \dot{n}_i \right) d^3\mathbf{r} + \int \left(\bar{\sigma}_{ij} v_{j,i} + \bar{h}_i \dot{n}_i \right) d^2\mathbf{r}, \tag{43}$$

where $\sigma'_{ij} = \sigma_{ij} - \sigma^e_{ij}$ and is not symmetric in general. The quantity $\bar{\sigma}_{ij}$ is a two-dimensional stress tensor:

$$\bar{\sigma}_{ij} = \int (\sigma_{ij} - p\delta_{ij}) \boldsymbol{\nu} \cdot d\mathbf{r}, \tag{44}$$

where p is the static pressure, and δ_{ij} is the Kronecker delta.

The total dissipation, given by Eq. (43), can be separated into symmetric and antisymmetric parts, i.e. $\sigma'_{ij} = \Pi_{ij} + \Gamma_{ij}$, $\bar{\sigma}_{ij} = \bar{\Pi}_{ij} + \bar{\Gamma}_{ij}$, and $v_{i,j} = A_{ij} + \omega_{ij}$. The symmetric and antisymmetric part of the velocity gradient in the fluid can be expressed as

$$A_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \quad \text{and} \quad \omega_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i}). \tag{45}$$

Thus $T\dot{s}$ can be rewritten as

$$T\dot{s} = \int \left\{ \Pi_{ij} A_{ij} + \Gamma_{ij} \left(\omega_{ji} - \Omega_{ji} \right) \right\} d^3\mathbf{r} + \int \left(\bar{\Pi}_{ij} \bar{A}_{ij} - \bar{\Gamma}_{ij} \Omega_{ji} \right) d^2\mathbf{r}, \tag{46}$$

where $\Omega_{ij} = n_i \dot{n}_j - \dot{n}_i n_j$ (the internal angular velocity), $\bar{\Gamma}_{ij} = \bar{h}_i n_j$, and $\bar{A}_{ij} = (\bar{v}_{i,j} + \bar{v}_{j,i})/2$. Note that Γ_{ij} and $\bar{\Gamma}_{ij}$ are the antisymmetric parts of the viscous stresses, i.e. the viscous torques.

2.2 Friction coefficients and viscous torques. We choose A_{ij}, $(\omega_{ij} - \Omega_{ij})$, $\bar{A}_{ij}$, and Ω_{ij} as fluxes; the forces conjugate to the fluxes are Π_{ij}, Γ_{ij}, $\bar{\Pi}_{ij}$, and $\bar{\Gamma}_{ij}$. These forces can be expressed by expansions in terms of the fluxes and Q_{ij}[20]; i.e.

$$\begin{aligned} \Pi_{ij} = {} & \eta A_{ij} + a_1 \left(Q_{ik} A_{kj} + A_{ik} Q_{kj} \right) + a_2 Q_{kl} Q_{kl} A_{ij} + a_3 Q_{kl} A_{lk} Q_{ij} \\ & + a_4 \left(Q_{ik} Q_{kl} A_{lj} + A_{ik} Q_{kl} Q_{lj} \right) + a_5 Q_{ik} A_{kl} Q_{lj} \\ & + b_1 \left\{ Q_{ik} \left(\omega_{kj} - \Omega_{kj} \right) - \left(\omega_{ik} - \Omega_{ik} \right) Q_{kj} \right\} \\ & + b_2 \left\{ Q_{ik} Q_{kl} \left(\omega_{lj} - \Omega_{lj} \right) - \left(\omega_{ik} - \Omega_{ik} \right) Q_{kl} Q_{lj} \right\}, \end{aligned} \tag{47}$$

$$\begin{aligned}
\Gamma_{ij} = {} & b_1\left(Q_{ik}A_{kj} - A_{ik}Q_{kj}\right) + b_2\left(Q_{ik}Q_{kl}A_{lj} - A_{ik}Q_{kl}Q_{lj}\right) \\
& + c_1\left\{Q_{ik}\left(\omega_{kj} - \Omega_{kj}\right) + \left(\omega_{ik} - \Omega_{ik}\right)Q_{kj}\right\} \\
& + c_2 Q_{kl}Q_{kl}\left(\omega_{ij} - \Omega_{ij}\right) \\
& + c_3\left\{Q_{ik}Q_{kl}\left(\omega_{lj} - \Omega_{lj}\right) + \left(\omega_{ik} - \Omega_{ik}\right)Q_{kl}Q_{lj}\right\} \\
& + c_4 Q_{ik}\left(\omega_{kl} - \Omega_{kl}\right)Q_{lj},
\end{aligned} \tag{48}$$

$$\begin{aligned}
\overline{\Pi}_{ij} = {} & \overline{\eta A}_{ij} + \bar{a}_1\left(\overline{Q}_{ik}\overline{A}_{kj} + \overline{A}_{ik}\overline{Q}_{kj}\right) + \bar{a}_2\overline{Q}_{kl}\overline{Q}_{kl}\overline{A}_{ij} + \bar{a}_3\overline{Q}_{kl}\overline{A}_{lk}\overline{Q}_{ij} \\
& + \bar{a}_4\left(\overline{Q}_{ik}\overline{Q}_{kl}\overline{A}_{lj} + \overline{A}_{ik}\overline{Q}_{kl}\overline{Q}_{lj}\right) + \bar{a}_5\overline{Q}_{ik}\overline{A}_{kl}\overline{Q}_{lj} \\
& + \bar{b}_1\left(\Omega_{ik}\overline{Q}_{kj} - \overline{Q}_{ik}\Omega_{kj}\right) + \bar{b}_2\left(\Omega_{ik}\overline{Q}_{kl}\overline{Q}_{lj} - \overline{Q}_{ik}\overline{Q}_{kl}\Omega_{lj}\right),
\end{aligned} \tag{49}$$

$$\begin{aligned}
\overline{\Gamma}_{ij} = {} & \bar{b}_1\left(\overline{Q}_{ik}\overline{A}_{kj} - \overline{A}_{ik}\overline{Q}_{kj}\right) + \bar{b}_2\left(\overline{Q}_{ik}\overline{Q}_{kl}\overline{A}_{lj} - \overline{A}_{ik}\overline{Q}_{kl}\overline{Q}_{lj}\right) \\
& + \bar{c}_1\left(\Omega_{ik}\overline{Q}_{kj} + \overline{Q}_{ik}\Omega_{kj}\right) + \bar{c}_2\Omega_{ik}\overline{Q}_{kl}\overline{Q}_{lj} \\
& + \bar{c}_3\left(\Omega_{ik}\overline{Q}_{kl}\overline{Q}_{lj} + \overline{Q}_{ik}\overline{Q}_{kl}\Omega_{lj}\right) + \bar{c}_4\overline{Q}_{ik}\Omega_{kl}\overline{Q}_{lj}.
\end{aligned} \tag{50}$$

We have been discussing the case of a constant-density or incompressible fluid; thus the divergence of the fluid velocity $v_{i,i}$ is zero. Finally, eqs. (47)–(50) reduce to the following formulae.[20]

$$\begin{aligned}
\Pi_{ij} = {} & A_1 S^2 n_k n_l A_{lk} n_i n_j + \left(\eta - a_1 S + A_3 S^2\right)A_{ij} \\
& + \left\{\left(\tfrac{3}{2}a_1 + B_1\right)S + \left(A_2 + B_2\right)S^2\right\}n_i n_k A_{kj} \\
& + \left\{\left(\tfrac{3}{2}a_1 - B_1\right)S + \left(A_2 - B_2\right)S^2\right\}A_{ik} n_k n_j \\
& + \left\{-\left(B_1 + C_1\right)S - \left(B_2 + C_2\right)S^2\right\}n_i N_j \\
& + \left\{-\left(B_1 - C_1\right)S - \left(B_2 - C_2\right)S^2\right\}N_i n_j,
\end{aligned} \tag{51}$$

$$\begin{aligned}
\overline{\Pi}_{ij} = {} & \overline{A}_1\overline{S}^2\bar{n}_k\bar{n}_l\overline{A}_{lk}\bar{n}_i\bar{n}_j + \left(\bar{\eta} - \bar{a}_1\overline{S} + \overline{A}_3\overline{S}^2\right)\overline{A}_{ij} \\
& + \left\{\left(\tfrac{3}{2}\bar{a}_1 + \overline{B}_1\right)\overline{S} + \left(\overline{A}_2 + \overline{B}_2\right)\overline{S}^2\right\}\bar{n}_i\bar{n}_k\overline{A}_{kj} \\
& + \left\{\left(\tfrac{3}{2}\bar{a}_1 - \overline{B}_1\right)\overline{S} + \left(\overline{A}_2 - \overline{B}_2\right)\overline{S}^2\right\}\overline{A}_{ik}\bar{n}_k\bar{n}_j \\
& + \left\{-\left(\overline{B}_1 + \overline{C}_1\right)\overline{S} - \left(\overline{B}_2 + \overline{C}_2\right)\overline{S}^2\right\}\bar{n}_i\overline{N}_j \\
& + \left\{-\left(\overline{B}_1 - \overline{C}_1\right)\overline{S} - \left(\overline{B}_2 - \overline{C}_2\right)\overline{S}^2\right\}\overline{N}_i\bar{n}_j,
\end{aligned} \tag{52}$$

$$\begin{aligned}
\Gamma_{ij} = {} & \left(2C_1 S + 2C_2 S^2\right)\left(n_i N_j - N_i n_j\right) \\
& + \left(-2B_1 S - 2B_2 S^2\right)\left(n_i n_k A_{kj} - A_{ik} n_k n_j\right),
\end{aligned} \tag{53}$$

$$\begin{aligned}
\overline{\Gamma}_{ij} = {} & \left(2\overline{C}_1\overline{S} + 2\overline{C}_2\overline{S}^2\right)\left(\bar{n}_i\overline{N}_j - \overline{N}_i\bar{n}_j\right) \\
& + \left(-2\overline{B}_1\overline{S} - 2\overline{B}_2\overline{S}^2\right)\left(\bar{n}_i\bar{n}_k\overline{A}_{kj} - \overline{A}_{ik}\bar{n}_k\bar{n}_j\right),
\end{aligned} \tag{54}$$

where $N_i = \dot{n}_i - n_k \omega_{ki}$ is the time rate of change of the director relative to the fluid rotation and $\bar{N}_i = \dot{\bar{n}}_i$. The viscous torques Γ and $\bar{\Gamma}$ are given by

$$\Gamma_v = \left(2C_1 S + 2C_2 S^2\right)\mathbf{n} \times \left(\dot{\mathbf{n}} - \tfrac{1}{2}\operatorname{rot}\mathbf{v} \times \mathbf{n}\right)$$
$$+\left(-2B_1 S - 2B_2 S^2\right)\mathbf{n} \times (\mathbf{n} : \mathbf{A}), \tag{55}$$
$$\bar{\Gamma}_v = \left(2\bar{C}_1\bar{S} + 2\bar{C}_2\bar{S}^2\right)\bar{\mathbf{n}} \times \dot{\bar{\mathbf{n}}} + \left(-2\bar{B}_1\bar{S} - 2\bar{B}_2\bar{S}^2\right)\bar{\mathbf{n}} \times (\bar{\mathbf{n}} : \bar{\mathbf{A}}). \tag{56}$$

In the equilibrium state, $\Pi_{ij} = 0$, $\bar{\Pi}_{ij} = 0$, and the viscous torques Γ_v and $\bar{\Gamma}_v$ are equal to Eqs. (18) and (19), respectively.

2.3 *Discussion*

A. Equations and unknowns

The final formulas to explain nematic liquid crystals are:

1) Eqs. (16) and (17), 2) $\Pi_{ij} = 0$ (Eq. (51)), 3) $\bar{\Pi}_{ij} = 0$ (Eq. (52)), 4) $\Gamma = \Gamma_v$ (Eqs. (18) and (55)), 5) $\bar{\Gamma} = \bar{\Gamma}_v$ (Eqs. (19) and (56)), and 6) $\operatorname{div}\mathbf{v} = 0$.

The number of equations is $2 + 3 + 2 + 2 + 2 + 1 = 12$. The unknowns of these equations are: 1) $S(r, t)$ and $\bar{S}(t)$, 2) $\mathbf{v}(r, t)$, 3) $\bar{\mathbf{v}}(t)$, 4) $\mathbf{n}(r, t)$, 5) $\bar{\mathbf{n}}(t)$, and 6) $p(r, t)$. The number of unknowns is then $2 + 3 + 2 + 2 + 2 + 1 = 12$. Because $\mathbf{n}$ is a unit vector, it involves two independent parameters. The numbers of equations and unknowns just agree with each other. From formula 1), the order parameters are determined; from 2) and 3), the velocities; from 4) and 5), the directors; and from 6), the pressure. However, it is difficult to solve these equations completely.

B. The case of $\mathbf{v} = 0$

In this case, eqs. (51)–(56) are very much simplified; i.e.

$$\Gamma_v = \left(2C_1 S + 2C_2 S^2\right)\mathbf{n} \times \dot{\mathbf{n}}, \tag{57}$$
$$\bar{\Gamma}_v = \left(2\bar{C}_1\bar{S} + 2\bar{C}_2\bar{S}^2\right)\bar{\mathbf{n}} \times \dot{\bar{\mathbf{n}}}, \tag{58}$$
$$\Pi_{ij} = \left\{-(B_1 + C_1)S - (B_2 + C_2)S^2\right\} n_i \dot{n}_j$$
$$+\left\{-(B_1 - C_1)S - (B_2 - C_2)S^2\right\} \dot{n}_i n_j = 0, \tag{59}$$
$$\bar{\Pi}_{ij} = \left\{-\left(\bar{B}_1 + \bar{C}_1\right)\bar{S} - \left(\bar{B}_2 + \bar{C}_2\right)\bar{S}^2\right\} \bar{n}_i \dot{\bar{n}}_j$$
$$+\left\{-\left(\bar{B}_1 - \bar{C}_1\right)\bar{S} - \left(\bar{B}_2 - \bar{C}_2\right)\bar{S}^2\right\} \dot{\bar{n}}_i \bar{n}_j = 0. \tag{60}$$

Equations (59) and (60) yield $\dot{n}_i \equiv 0$. This implies that the flow is generated by time rate of change of the director.

C. Example

We write down the fundamental equations for homogeneously aligned nematic liquid crystals as an example. The director, the easy axis, velocity, and electric field are taken as follows:

$$\mathbf{n} = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \cos\phi \\ 0 \\ \sin\phi \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}, \quad \boldsymbol{\nu} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{61}$$

Substituting Equation (61) into Eqs. (16)–(19) and (51)–(54), the following six equations are obtained:

$$\begin{aligned} &-\frac{1}{2}\Delta\alpha\sin^2\theta E^2 + f(S) + \left(L_1'S\cos^2\theta + L_2'S\sin^2\theta\right)\left(\frac{\partial\theta}{\partial z}\right)^2 \\ &\quad -\left(L_1'' + L_2''\sin^2\theta\right)\frac{\partial^2 S}{\partial z^2} - 2L_2''\sin\theta\cos\theta\left(\frac{\partial S}{\partial z}\right)^2 \\ &\quad -\tfrac{1}{2}\left(L_1''' - L_2'''\right)\sin\theta\cos\theta\frac{\partial\theta}{\partial z}\frac{\partial S}{\partial z} = 0, \end{aligned} \tag{62}$$

$$\begin{aligned} &-\left(\frac{1}{2}\Delta\beta + \Delta\gamma\bar{S}\right)\cos^2(\bar{\theta} - \phi) \\ &\quad +\left(L_1'' + L_2''\sin^2\bar{\theta}\right)\overline{\frac{\partial S}{\partial z}}\tfrac{1}{2}L_1'''\bar{S}\sin\bar{\theta}\cos\bar{\theta}\overline{\frac{\partial\theta}{\partial z}} = 0, \end{aligned} \tag{63}$$

$$\begin{aligned} &\left(L_1'S^2 - L_3'S^2\right)\sin\theta\cos\theta\left(\frac{\partial\theta}{\partial z}\right)^2 - \left(L_1'S^2\cos^2\theta + L_3'S^2\sin^2\theta\right)\frac{\partial^2\theta}{\partial z^2} \\ &\quad + L_2''\sin\theta\cos\theta\left(\frac{\partial S}{\partial z}\right)^2 + \Delta\alpha S\sin\theta\cos\theta E^2 \\ &\quad = \left(2C_1S + 2C_2S^2\right)\frac{\partial\theta}{\partial t} + \left(-2B_1S - 2B_2S^2\right)\sin\theta\cos\theta\frac{\partial v}{\partial z}, \end{aligned} \tag{64}$$

$$\begin{aligned} &\left(L_1'\bar{S}^2\cos^2\bar{\theta} - L_3'\bar{S}^2\sin^2\bar{\theta}\right)\overline{\frac{\partial\theta}{\partial z}} + \left(\Delta\beta\bar{S} + \Delta\gamma\bar{S}^2\right)\sin(\bar{\theta} - \phi)\cos(\bar{\theta} - \phi) \\ &\quad -\tfrac{1}{2}\left(L_1'''\sin\theta\cos\theta + L_2'''\cos\theta\right)\bar{S}\overline{\frac{\partial S}{\partial z}} \\ &\quad = \left(2\bar{C}_1\bar{S} + 2\bar{C}_2\bar{S}^2\right)\frac{\partial\bar{\theta}}{\partial t} + \left(-2\bar{B}_1\bar{S} - 2\bar{B}_2\bar{S}^2\right)\sin\bar{\theta}\cos\bar{\theta}\overline{\frac{\partial v}{\partial z}}, \end{aligned} \tag{65}$$

$$A_1S^2\sin^2\theta\cos^2\theta\frac{\partial v}{\partial z} - \left(-2B_1S - 2B_2S^2\right)\sin\theta\cos\theta\frac{\partial\theta}{\partial t} = 0, \tag{66}$$

$$\bar{A}_1\bar{S}^2\sin^2\bar{\theta}\cos^2\bar{\theta}\overline{\frac{\partial v}{\partial z}} - \left(-2\bar{B}_1\bar{S} - 2\bar{B}_2\bar{S}^2\right)\sin\bar{\theta}\cos\bar{\theta}\frac{\partial\bar{\theta}}{\partial t} = 0. \tag{67}$$

Solving the above equations, the six unknowns $\mathbf{n}$, $\bar{\mathbf{n}}$, S, $\bar{S}$, $\mathbf{v}$, and $\bar{\mathbf{v}}$ (or θ, $\bar{\theta}$, S, $\bar{S}$, $\mathbf{v}$, and $\bar{\mathbf{v}}$) can be obtained. If we assume S is constant, Eqs. (62) and (63) give the order parameter, and Eqs. (64)–(67) become well known formulae; i.e.,

$$\begin{aligned}(k_1 - k_3)\sin\theta\cos\theta\left(\frac{\partial\theta}{\partial z}\right)^2 &- \left(k_1\cos^2\theta + k_3\sin^2\theta\right)\frac{\partial^2\theta}{\partial z^2}\\ &+ \Delta\epsilon\sin\theta\cos\theta E^2 = \gamma_1\frac{\partial\theta}{\partial t} + \gamma_2\sin\theta\cos\theta\frac{\partial v}{\partial z},\end{aligned} \tag{68}$$

$$\begin{aligned}\left(\bar{k}_1\cos^2\bar{\theta} - \bar{k}_3\sin^2\bar{\theta}\right)\frac{\partial\bar{\theta}}{\partial z} &+ \Delta\pi\sin(\bar{\theta} - \phi)\cos(\bar{\theta} - \phi)\\ &= \bar{\gamma}_1\frac{\partial\bar{\theta}}{\partial t} + \bar{\gamma}_2\sin\bar{\theta}\cos\bar{\theta}\frac{\partial\bar{v}}{\partial z},\end{aligned} \tag{69}$$

$$\alpha_1\sin\theta\cos\theta\frac{\partial v}{\partial z} = \gamma_2\frac{\partial\theta}{\partial t}, \tag{70}$$

$$\bar{\alpha}_1\sin\bar{\theta}\cos\bar{\theta}\frac{\partial\bar{v}}{\partial z} = \bar{\gamma}_2\frac{\partial\bar{\theta}}{\partial t}. \tag{71}$$

The initial state of the director $\mathbf{n}_0$ is given by putting $E = 0$ and $v = 0$; $\mathbf{n}_0$ is then

$$\mathbf{n}_0 = \begin{pmatrix}\cos\phi\\ 0\\ \sin\phi\end{pmatrix}. \tag{72}$$

Using Eq. (72) as an initial condition, we can obtain the director transient upon applying an electric field by solving Eqs. (68)–(71).

III. FUTURE POINT OF VIEW

In this paper, we have restricted our attention to uniaxial, incompressible nematic liquid crystals. As mentioned in the introduction, translational and biaxial order parameters and, furthermore, the density should be introduced in order to explain the smectic, re-entrant nematic, and discotic mesophases.[21]

The uniaxial order is described by

$$Q_{ij} = \frac{3}{2}S\left(n_i n_j - \frac{1}{3}\delta_{ij}\right) = S\begin{pmatrix}-\frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 1\end{pmatrix}, \tag{73}$$

where the director orients parallel to the z-axis. The biaxial order could be described by the tensor

$$Q_{ij}^{b} = \begin{pmatrix} -\frac{1}{2}S + \frac{1}{2}P & & \\ & -\frac{1}{2}S - \frac{1}{2}P & \\ & & S \end{pmatrix}. \tag{74}$$

Using a second director **m** (**m** ⊥ **n**, **n** ⊥ **l**, and **m** ⊥ **l**) this expression can be written as follows[22]:

$$Q_{ij}^{b} = \tfrac{3}{2}S\left(n_i n_j - \tfrac{1}{3}\delta_{ij}\right) + \tfrac{1}{2}P\left(m_i m_j - l_i l_j\right). \tag{75}$$

The quantity P is a second orientational order parameter.

The translational order[23] is described by

$$\Psi_i = \Psi_0 \exp\left(i q_0 x_i\right). \tag{76}$$

The density as an order parameter is expressed by

$$\rho = \rho' - \rho_0. \tag{77}$$

The quantities ρ' and ρ_0 are an actual density and the optimum density, respectively.[24]

In order to describe the various liquid crystaline phases, the free energy should be expanded in terms of order parameters Q_{ij}^{b}, Ψ_i, and ρ. This leads to very complex formulae. However, almost all the phenomena observed in liquid crystals would be explained by these equations in a unified manner. For example, smectics could be described using Q_{ij}^{b} and Ψ_i; re-entrant mesophases could be described using ρ and Ψ_i, etc.

IV. CONCLUSION

Several phenomena in nematic liquid crystals which had been discussed individually, have been explained in a unified manner using the free energy expanded in powers of the uniaxial orientational order parameter. Surface phenomena are also explained by introducing the surface energy. The director **n**, the scalar order parameter S, and the velocity are completely obtained by solving the fundamental equations.

Further development will be necessary to describe other phenomena in liquid crystals.

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